MATH2050C Selected Solutions to Assignment 2

Section 2.4 no. 14, 15, 17.

Section 2.4

(14) Show that there exists $n \in \mathbb{N}$ such that $1/2^n < y$ for any $y > 0$.

Solution It is equivalent to showing there is some n such that $2^n > 1/y$. We may simply imitate the proof of the Archimedean Property 2.4.3. Or, we use Archimedean property to find some n such that $1/y < n$ and then use the binomial formula $2^n = (1+1)^n \geq 1+n > n$.

(17) Show that $u^3 = 2$ is solvable.

Solution Let $E = \{x \in \mathbb{R} : \ x^3 < 2\}$. Clearly E is nonempty. For $x \in E, x^3 < 2 < 8$. From $x^3 - 8 = (x + 2x + 4)(x - 2) < 0$ we see that $x < 2$, that is E is bounded above by 2. By Order Completeness $z = \sup E \in \mathbb{R}$ exists. We shall show that $z^3 = 2$ by excluding $z^3 > 2$ and $z^3 < 2$. In case $z^3 > 2$, for some $\varepsilon \in (0,1)$ to be chosen later, $(z-\varepsilon)^3 = z^3 - 3\varepsilon z^2 + 3\varepsilon^2 z - \varepsilon^3 > z^3 - 3\varepsilon z^2 - \varepsilon = (z^3-2) - \varepsilon(3z^2+1) + 2$. If we further choose ε so that $(z^3-2)/(3z^2+1) \geq \varepsilon$, then $(z-\varepsilon)^3 > 2$, contradicting the definition of z. Next, in case $z^3 < 2$, $(z+\varepsilon)^3 = z^3 + 3\varepsilon z^2 + 3\varepsilon^2 z + \varepsilon^3$. For $\varepsilon \in (0,1)$, $z^3 + 3\varepsilon z^2 + 3\varepsilon^2 z + \varepsilon^3 \leq (z^3-2) + 2 + 3\varepsilon z^2 + 3\varepsilon z + \varepsilon$. Hence if we further choose ε such that

$$\varepsilon(3z^2 + 3z + 1) < 2 - z^3 \ ,$$

we have $(z+\varepsilon)^3 < 2$, again contradicting the definition of z.

Supplementary Problems

(1) Prove the Nested Interval Property: Let $[a_n, b_n]$ satisfies $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ for all $n \geq 1$. Show that there is $x \in \mathbb{R}$ such that $x \in [a_n, b_n]$ for all $n \geq 1$. Hint: Use the order completeness property.

Solution Since the intervals are nested, $a_n \leq a_{n+1} \leq \cdots \leq b_1$. The set $\{a_n\}, n \geq 1$, is bounded above. By the order completeness property, $x \equiv \sup\{a_n\}$ exists. If we could prove $x \leq b_n$ for all n, then $x \in [a_n, b_n]$ for all n. Suppose not, that is, $x > b_{n_0}$ for some n_0. Since b_{n_0} is an upper bound for all $a_n, n \geq n_0$, and x is the supremum of a_n's, $b_{n_0} \geq x$. We have arrived at $x > x$, contradiction holds.

(2) Find the decimal representation of the numbers 0.502 and 1/7.

(3) Show that there are infinitely many rational and irrational numbers lying between two distinct numbers.

Solution It suffices to consider $0 < x < y$. By Proposition 2.2, there is $r_1 \in \mathbb{Q}$ such that $x < r_1 < y$. Applying the same prop to x and r_1 we get $r_2 \in \mathbb{Q}$ satisfying $x < r_2 < r_1 < y$. Keep doing this we get $\{r_k\}, k \geq 1$ in $\mathbb{Q}$ lying between x and y. Similar we get irrationals lying between x and y.

(4) Show that the cardinal number of any interval is equal to the cardinal number of $\mathbb{R}$.

Solution Since every interval I contains an open interval (a,b), $|(a,b)| \leq |I| \leq \mathbb{R}$. If one could show that $|(a,b)| = |(0,1)|$, from Proposition 2.7 we conclude $|I| \geq |(a,b)| = |\mathbb{R}|$. Hence $|I| = |\mathbb{R}|$

by Schroder-Bernstein. Now the map $f(x) = a + x(b - a)$ maps $(0, 1)$ bijectively to (a, b), done.

(5) Show that $|\mathbb{R}^2| = |\mathbb{R}|$. Recall that $\mathbb{R}^2 = \{(x, y) : \ x, y \in \mathbb{R}\}$.

Solution The map $(x, y) \mapsto (\tan \pi/2(x - 1/2), \tan \pi/2(y - 1/2))$ is bijective from $(0, 1)^2$ to $\mathbb{R}^2$. It suffices to show $|(0, 1)^2| = |(0, 1)|$. The idea is to use the decimal representation. For $z \in (0, 1), z = 0.a_1a_2a_3a_4a_5a_6 \cdots$, we associate it to $x = 0.a_1a_2a_3 \cdots$ and $y = 0.a_2a_4a_6 \cdots$ to get a surjective map from $(0, 1)$ to $(0, 1)^2$. Hence $|(0, 1)| \geq |(0, 1)^2|$. On the other hand, the set $(0, 1) \times \{1/2\}$ which is just a copy of $(0, 1)$ is a subset of $(0, 1)^2$, hence $|(0, 1)^2| \geq |(0, 1) \times \{1/2\}| = |(0, 1)|$. By Schroder-Bernstein Theorem $|(0, 1)^2| = |(0, 1)|$.

Note. By induction $|\mathbb{R}^n| = |\mathbb{R}|$.

(6) A real number is called an algebraic number if it is a root of some equation $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ with integral coefficients. Show that the set of all algebraic numbers is a countable set containing all rational numbers and numbers of the form $a^{1/k}, a > 0, k \geq 1$.

Solution Let A_n be the set consisting of all equations of degree equal to n, that is, $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ where $a_j \in \mathbb{Q}$. A_n is countable. Let Z_n be the set of the roots of the equations from A_n. Since each equation has at most n-many real roots, Z_n is also countable. Now the set of all algebraic numbers are $\bigcup_n Z_n$, hence it is also countable.

Note It shows that there are uncountably many transcendental numbers.